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Local Cohomology and Symbolic Power of Modules

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Abstract

Let I be an ideal of a Noetherian ring R,and let N be a non-zero finitely generated R-module. In this paper, the main results are a relationship between the vanishing of the local cohomology modules $H_I^i(N)$, and a comparison of the topologies$I$ defined by the $\{\overline{I^n N}\}n \geq 0$ and $\{S(\overline{I^n N})\}n \geq 0$.

Key words: Asymptotic prime, Integral closure, Ideal topology, Local cohomology

Introduction

Let R denote a commutative Noetherian ring, I an ideal of R, and N a non-zero finitely generated R-module. $\bar{I}$ denotes the integral closure of $\bar{I}$, $i.e.$, $\bar{I}$ is the ideal of R consisting of all elements $x \epsilon R$, which satisfy the equation $x_n + r_1 x_{n-1} + \cdots + r_n$, where $r_i \epsilon I_i$, $i = 1,...,n$. The concept of integral closure of an ideal of a commutative Noetherian ring (with identity), developed by D. G. Northcott and D. Rees (1985), is fundamental to a considerable body of recent and current research both in commutative algebra and algebraic geometry. Moreover, the notion of integral closures of ideals of R relative to a Noetherian R-module N was initiated by R.Y. Sharp et al. (1990). An element x R is said to be integrally dependent on I relative to N if there exists a positive integer n such that $x^n \subseteq \sum_{i=1}^{n} x^{n-i} I^i N$. Then the set

$$\overline{IN} = \{x \in R;\ x \text{ is integrally dependent on } I \text{ relative to } N\ \}$$

is an ideal of R, called the integral closure of I relative to N , in the case N=R, $\overline{IN}$ is the classical integral closure $\bar{I}$ of I. It is clear that $I \subseteq \overline{IN}$. I is considered as integrally closed relative to N if $I = \overline{IN}$. For any multiplicatively closed subset S of R, $S(\overline{IN})$ is the union of $(\overline{IN}:_R s)$, where s varies in S.

For any ideal I of R and any R-module N, the i-th local cohomology module of N with respect to I is defined by

$$H_I^i(N) = \varinjlim_{n \epsilon \mathbb{N}_0} Ext_R^i(R/I^n.\ N)$$

The reader is referred to (Brodmann and Sharp, 2012) for basic properties of local cohomology modules.
In the second section, the linearly equivalence between the topologies defined by $\{S(\overline{I^n N})\}_{n \geq 0}$ and $\{\overline{I^n N}\ \}_{n \geq 0}$ is studied in the terms of vanishing of the top local cohomology module $H_I^{\dim N}(N)$. The main result in this section is thefollowing:

Theorem 1.1. *Let R be a Noetherian ring, N a finitely generated R-module, and I an ideal of R. Then, for any multiplicatively closed subset S of R, the following are equivalent:*

(i) $S \subseteq R \setminus \{p \in \overline{A^*}\,(I, N)\}$.
(ii) For all integers $k \geq 0$, $S(\overline{I^k N}) \subseteq \overline{I^k N}$ *(hence equality holds).*
(iii) There exists an integer $h \geq 0$ *such that for all integers* $k \geq 0$, $S(\overline{I^{k+h}} N) \subseteq \overline{I^k N}$.

One of our tools for proving Theorem 1.1 is the following set, called quintasymptotic prime ideals of I with respect to N.

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$\overline{Q^*}$ $(I,\ N)$ = {p ∈ Spec(R); there exists a q ∈ $mAss_{\hat{R}_p}\ \widehat{Np}$ such that Rad$(I\widehat{Rp}\ +\ q)\ =\ p\widehat{Rp}$} In the case N = R, this set was studied by McAdam (1987). The set of asymptotic prime ideals of I with respect to N is defined as:

$\overline{A^*}(I,N) = \{q \cap R;\ q\ \in\ \overline{Q^*}(uR[It,u], N\ [It,u])\}$

in which, $R[It,u]$ and $N[It,u]$, are respectively the Rees ring of R with respect to I, and Rees module of N with respect to I.

The concepts of $\overline{Q^*}(I, N)$ and $\overline{A^*}$ (I, N) have been studied by Ahn (1995).

The final section discusses some relationships between 0-th modules of local cohomology and the symbolic and ordinary powers. The following Theorem is the main result of this section:

Theorem 1.2. *Let R be a Noetherian ring, N a finitely generated R-module, and I an ideal of R. Then the following statements hold:*

(i) Let J be a second ideal of R. Then there exists a multiplicatively closed subset S of Rsuch that $H_J^0(N/I_nN) = S(I^nN)/I^nN$ *for all integers* $n \geq 0$
(ii) Let S be a multiplicatively closed subset of R. Then there exists an ideal J of R, such that $H_J^0(N/I_nN) = S(I^nN)/I^nN$*for all integers* $n \geq 0$.

Local Cohomology and Ideal Topology

The purpose of this section is to show the linear equivalence between the topologies defined by by $\{S(\overline{I^nN})\}n \geq 0$ and $\{\overline{I^nN}\}_{n\geq0}$ in terms of vanishing of the top local cohomology module $H_I^{\dim N}(N)$. We start with the following remark:

Let R be a Noetherian ring, N a finitely generated R-module, and I an ideal of R. Then for any multiplicatively closed subset S of R, we have the following equality:

$$S(\overline{I^nN}) = S^{-1}(\overline{I^nN}) \cap R$$

The following Theorem was proved by McAdam (1987), in the case $N = R$. The proof can be easily carried over to a module, so we omit the proof.

Theorem 2.1. *Let R be a Noetherian ring, N a finitely generated R-module, and I an ideal of R. Then for any multiplicatively closed subset S of R, consider the following:*

(i) $S\ \subseteq\ R \setminus \cup \{p\ \in\ \overline{A^*}\ (I,N)\}$.
(ii) For all integers $k\ \geq\ 0$, $S(\overline{I^kN}) \subseteq \overline{I^kN}$ *(hence equality holds).*
(iii) There exists an integer $h \geq 0$ *such that for all integers* $k \geq\ 0$, $S(\overline{I^{k+h}N}) \subseteq \overline{I^kN}$.

Then we have the implications: $(i) \Rightarrow (ii) \Rightarrow (iii)$.

The following Corollary is an immediate consequence of the above-mentioned Theorem.

Corollary 2.2. *Let R be a Noetherian ring, I an ideal of R, and N a finitely generated R- module, such that* $\overline{A^*}$(I, N)= $mAss_RN/IN$ *Then the topologies defined by* $\{S(\overline{I^nN})\}_{n\geq0}$ and $\{\overline{I^nN}\}_{n\geq0}$ *are linearly equivalent.*

Theorem 2.3. *Let* (R, m) *be a Noetherian local ring, N a finitely generated R-module of dimension d and I an ideal of R. Consider the following conditions:*

(i) There exists a multiplicatively closed subset S of R such that $\mathrm{m} \cap S\ \neq\ \phi$, *and that the the topologies defined by* $\{S(\overline{I^nN})\}_{n\geq0}$ and $\{\overline{I^nN}\}_{n\geq0}$ *are linearly equivalent.*
$(ii) H_I^d(N) = 0$.

Then, (i) *implies* (ii)*, and the conditions are equivalent, if N is quasi-unmixed.*

Proof. The implication $(i) \Rightarrow (ii)$ follows easily from (Naghipour and Schenzel, 2018, Theorem 4.1).

Assume N is quasi-unmixed. In order to prove the conclusion $(ii) \Rightarrow (i)$, let $S = R \setminus \cup\{\mathfrak{p} \in$ (A^*)¯(I, N)$\}$. In view of 2.1, the topologies defined by $\{S(\overline{I^n N})\}_{n\geq 0}$ and $\{\overline{I^n N}\}_{n\geq 0}$ are linearly equivalent. Now, it is enough to show that $\mathfrak{m} \cap S \neq \phi$ suppose the contrary that $\mathfrak{m} \cap S = \phi$ .Then $\mathfrak{m} \epsilon \overline{A^*}(I, N)$. Then by the definition and invoking (Ahn, 1995, Proposition 3 8), $\mathfrak{m} \epsilon \overline{Q^*}(I, N)$. So there exists $\mathfrak{p} \in mAss_{\hat{R}}\, \hat{N}$, such that $\mathrm{Rad}(I\hat{R} + p) = m\hat{R}$. As N is quasi-unmixed, dim $\hat{R}/I\hat{R} + q = 0$, for some $\mathfrak{q} \epsilon\, mAss_{\hat{R}}\, \hat{N}$ such that dim $\hat{R}/\mathfrak{q} = d.$. It follows by using (Divaani-Aazar and Schenzel, 2001, corollary 3 4) $H_I^d(N) \neq 0$ which is a contradiction.

0-Th Module of Local Cohomology and Symbolic Power

For any multiplicatively closed subset S of R, the n-th S-symbolic power of I with respect to N, denoted by $S(I^n N)$, is defined to be the union of $I^n N :_N s$ where s varies in S. The S-symbolic filtration $S(I^n N)_{n\geq 0}$ induces a topology on N called the S-symbolic topology. The purpose of this section is to show the relationship between certain 0-th modules of local cohomology and (the quotient of) symbolic and ordinary powers. We begin with the following Lemma.

Lemma 3.1. Let R be a Noetherian ring, N a finitely generated R-module, and I an ideal of R. Let M be a R-submodule of N. Then, we have the following equation*:*

H_I^0 $(N/M) = M :_N <I>/M$

Proof. This assertion follows easilythe definitions.

Now we state and prove the main results of this section.

Proposition 3.2. Let R be a Noetherian ring, N a finitely generated R-module, and I and ideal of R. Then for each n∈ ℕ, the following conditions are equivalent:

(i) $I^n N = S(I^n N)$.
$(ii) H_q^0$ $(N/I^n N) = 0$ *for all* q ∈ Spec(R) *such that* q ∩ $S = \varphi$.
$(iii) H_q^0$ $(N_q/I^n N_q = 0$ *for all* q ∈ Spec(R) *such that* q ∩ $S = \varphi$.

Proof. Let $n\in \mathbb{N}$. Then, $I^n N = S(I^n N)$, if, and only if, $S \subset R \setminus \cup\{\mathrm{p} \in \mathrm{Ass}_R\ N/I^n N\}$.

if, and only if, $q \notin Ass_R$ $N/I^n N$ for all $q \in (R)$ with $q \cap S \neq \phi$, that is, if and only if, $(N_q/I^n N_q) > 0$ for all q∈Spec(R) with $q \cap S \neq \phi$. Therefore, theconsequence immediatelyfollows (Brodmann and Sharp, 2012,Proposition6.2.7).

Theorem 3.3. Let R be a Noetherian ring, N a finitely generated R-module, and I an ideal of R. Then the following statements hold:

(i) Let J be a second ideal of R. Then, there exists a multiplicatively closed subset S of R such that H_J^0 $(\mathrm{N/I_n N}) = \mathrm{S(I^n N)/\ I^n N}$ for all integers $n \geq 0$.
(ii) Let S be a multiplicatively closed subset of R. Then, there exists an ideal J of R, such that H_J^0 $(\mathrm{N/I_n N}) = \mathrm{S(I^n N)/\ I^n N}$ for all integers $n \geq 0$.

Proof. In order to prove (i), using 3.1 suffices to show that there exists a multiplicatively closed subset S of R such that $I^n N :_N <J>= S(I^n N)$ for all integers $n \geq 0$. Let $Ass_R\ N/I^n N = p_1 \cap \cdots \cap p_l \cap \cdots \cap p_m$ for some $n \in \mathbb{N}_0$, such that $J \subseteq p_i$ *if,* and only if, $i \in \{1 \cdots l\}$. Pick $x \in J$ such that $x \notin \cup_{i=l+1}^m p_i$. Let $S = \{x\}_{i\geq 0}$. Then it is not hard to see that $I^n N :_N < J >= S(I^n N)$.

In order to prove (ii), it is sufficient to show that there exists an ideal J of R such that

$I^n N :_N < J >= S(I^n N)$. for all $n \geq 0$ Let $\{p_1 \cdots p_r\} \subseteq Ass_R\ N/I^n N$ for some $n \in \mathbb{N}_0$, such that $p_i \cap S \neq \phi$ for all $1 \leq i \leq r$. Set $J = \cap_{i=1}^r p_i$. It is not hard to see that $I^n N :_N < J >= S(I^n N)$**.**

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